Corrigé de l'exercice 1

Extrema sous contrainte en dimension 2 - corrige detaille

▶1. A - Facile (contrainte lineaire)

Etape 0 - Rappel

$$f(x,y) = x^2 + y^2 + 2x$$

$$g(x,y) = -2x + y - 2 = 0 \quad \Longleftrightarrow \quad -2x + y = 2$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$L(x,y,\lambda) = x^2 + y^2 + 2x - \lambda\left(-2x + y - 2\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x + 2, \qquad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = -2, \qquad \frac{\partial g}{\partial y} = 1$$

$$\nabla f(x,y) = (2x+2,\, 2y)\,, \qquad \nabla g(x,y) = (-2,\, 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2x + 2 = \lambda \cdot -2 \\ 2y = \lambda \\ -2x + y = 2 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-2\lambda - 2}{2}, \qquad y = \frac{\lambda}{2}$$

$$-2 \cdot \frac{-2\lambda - 2}{2} + \frac{\lambda}{2} = 2$$

$$\frac{(5)\lambda - (-4)}{2} = 2 \Longrightarrow \lambda = 0$$

$$\Longrightarrow (x,y) = (-1,\, 0)$$

Etape 5 - Valeurs et conclusion

$f(-1,0) = -1$

Sur la droite de contrainte, cette valeur est le minimum global ; il n'y a pas de maximum global.

$$\boxed{\text{Finalement : } (x,y) = (-1,0)\,,\ f_{\min} = -1,\ \text{pas de maximum global.}}$$

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = 2 + 2x$$
$$\phi(x) = f(x, 2+2x) = 5x^2 + 10x + 4$$
$$\phi'(x) = 10x + 10 = 0 \Longrightarrow x_* = -1,\ y_* = 0$$
$$\phi(x_*) = f_{\min} = -1$$

▶2. B - Facile (variante lineaire)

Etape 0 - Rappel

$$f(x,y) = x^2 + y^2 - 4x - 4y - 3$$
$$g(x,y) = 3x + y + 2 = 0 \quad \Longleftrightarrow \quad 3x + y = -2$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$
$$L(x,y,\lambda) = x^2 + y^2 - 4x - 4y - 3 - \lambda\,(3x + y + 2)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x - 4, \qquad \frac{\partial f}{\partial y} = 2y - 4$$
$$\frac{\partial g}{\partial x} = 3, \qquad \frac{\partial g}{\partial y} = 1$$
$$\nabla f(x,y) = (2x-4,\ 2y-4)\,, \qquad \nabla g(x,y) = (3,\ 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2x - 4 = \lambda \cdot 3 \\ 2y - 4 = \lambda \\ 3x + y = -2 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{3\lambda + 4}{2}, \qquad y = \frac{\lambda + 4}{2}$$

$$3 \cdot \frac{3\lambda + 4}{2} + \frac{\lambda + 4}{2} = -2$$

$$\frac{(10)\lambda - (-16)}{2} = -2 \Longrightarrow \lambda = -2$$

$$\Longrightarrow (x, y) = (-1,\ 1)$$

Etape 5 - Valeurs et conclusion

$$f\,(-1, 1) = -1$$

Sur la droite de contrainte, cette valeur est le minimum global ; il n'y a pas de maximum global.

$$\boxed{\text{Finalement : } (x, y) = (-1, 1)\,,\ f_{\min} = -1,\ \text{pas de maximum global.}}$$

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = -2 - 3x$$

$$\phi(x) = f\,(x, -2 - 3x) = 10x^2 + 20x + 9$$

$$\phi'(x) = 20x + 20 = 0 \Longrightarrow x_* = -1,\ y_* = 1$$

$$\phi(x_*) = f_{\min} = -1$$

▶3. C - Moyen (cercle)

Etape 0 - Rappel

$$f(x, y) = -2x + y + 3$$

$$g(x, y) = x^2 + y^2 - 5 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = -2x + y + 3 - \lambda\left(x^2 + y^2 - 5\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = -2, \qquad \frac{\partial f}{\partial y} = 1$$

$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = 2y$$

$$\nabla f(x, y) = (-2,\ 1)\,, \qquad \nabla g(x, y) = (2x,\ 2y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x, y) = 0 \end{cases}$$

$$\begin{cases} -2 = 2\lambda x \\ 1 = 2\lambda y \\ x^2 + y^2 = 5 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-2}{2\lambda}, \qquad y = \frac{1}{2\lambda}$$

$$\left(\frac{-2}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 5$$

$$\frac{5}{4\lambda^2} = 5 \Longrightarrow \lambda^2 = \frac{1}{4} \Longrightarrow \lambda = \pm\frac{1}{2}$$

$$\lambda = \frac{1}{2} \Rightarrow (x, y) = (-2,\, 1)\,, \qquad \lambda = -\frac{1}{2} \Rightarrow (x, y) = (2,\, -1)$$

Etape 5 - Valeurs et conclusion

$$f\,(-2, 1) = 8$$

$$f\,(2, -1) = -2$$

$$\boxed{\text{Finalement : } f_{\min} = -2 \text{ et } f_{\max} = 8.}$$

Etape 6 - Methode alternative

Parametrisation du cercle.

$$x = \sqrt{5}\cos t, \qquad y = \sqrt{5}\sin t$$

$$f(t) = \sqrt{5}\,(-2\cos t + 1\sin t) + 3$$

$$|-2\cos t + 1\sin t| \leq \sqrt{5} = \sqrt{5}$$

$$\Longrightarrow -2 \leq f(t) \leq 8$$

▶**4. D - Difficile (ellipse generale + quadratique)**

Etape 0 - Rappel

$$f(x, y) = x^2 + y^2$$

$$g(x, y) = 2x^2 + 3xy + 2y^2 - 7 = 0$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$
$$L(x,y,\lambda) = x^2 + y^2 - \lambda\left(2x^2 + 3xy + 2y^2 - 7\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y$$
$$\frac{\partial g}{\partial x} = 4x + 3y, \qquad \frac{\partial g}{\partial y} = 3x + 4y$$
$$\nabla f = (2x,\ 2y), \qquad \nabla g = (4x + 3y,\ 3x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 2x = \lambda(4x + 3y) \\ 2y = \lambda(3x + 4y) \\ 2x^2 + 3xy + 2y^2 = 7 \end{cases}$$

Etape 4 - Resolution pas a pas

Somme des deux equations : $2(x+y) = \lambda(4+3)(x+y)$

Difference : $2(x-y) = \lambda(4-3)(x-y)$

$\Longrightarrow (2 - (4+3)\lambda)(x+y) = 0, \quad (2 - (4-3)\lambda)(x-y) = 0$

(i) $x = y = t \Rightarrow (4+3)t^2 = 7 \Rightarrow t = \pm 1$

(ii) $x = -y = t \Rightarrow (4-3)t^2 = 7 \Rightarrow t = \pm\sqrt{7}$

$\mathcal{C} = \left\{(1,\ 1), (-1,\ -1), \left(\sqrt{7},\ -\sqrt{7}\right), \left(-\sqrt{7},\ \sqrt{7}\right)\right\}$

Etape 5 - Valeurs et conclusion

$$f(1,1) = 2$$
$$f(-1,-1) = 2$$
$$f\left(\sqrt{7}, -\sqrt{7}\right) = 14$$
$$f\left(-\sqrt{7}, \sqrt{7}\right) = 14$$

Finalement : $f_{\min} = 2,\ f_{\max} = 14.$

Etape 6 - Methode alternative

Autre approche : diagonaliser la contrainte via u=x+y et v=x-y.

$$2x^2 + 3xy + 2y^2 = \frac{4+3}{4}u^2 + \frac{4-3}{4}v^2, \qquad f = \frac{u^2 + v^2}{2}$$

▶5. E - Plus difficile (ellipse generale + quadratique riche)

Etape 0 - Rappel

$$f(x,y) = 3x^2 - 2xy + 3y^2$$
$$g(x,y) = 2x^2 + 1xy + 2y^2 - 5 = 0$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$
$$L(x,y,\lambda) = 3x^2 - 2xy + 3y^2 - \lambda\left(2x^2 + 1xy + 2y^2 - 5\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 6x - 2y, \qquad \frac{\partial f}{\partial y} = -2x + 6y$$
$$\frac{\partial g}{\partial x} = 4x + 1y, \qquad \frac{\partial g}{\partial y} = 1x + 4y$$
$$\nabla f = (6x - 2y,\, -2x + 6y), \qquad \nabla g = (4x + 1y,\, 1x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 6x - 2y = \lambda(4x + 1y) \\ -2x + 6y = \lambda(1x + 4y) \\ 2x^2 + 1xy + 2y^2 = 5 \end{cases}$$

Etape 4 - Resolution pas a pas

Somme des deux equations : $4(x+y) = \lambda(4+1)(x+y)$

Difference : $8(x-y) = \lambda(4-1)(x-y)$

$\Longrightarrow (4 - (4+1)\lambda)(x+y) = 0, \quad (8 - (4-1)\lambda)(x-y) = 0$

(i) $x = y = t \Rightarrow (4+1)t^2 = 5 \Rightarrow t = \pm 1$

(ii) $x = -y = t \Rightarrow (4-1)t^2 = 5 \Rightarrow t = \pm\sqrt{\frac{5}{3}}$

$$\mathcal{C} = \left\{(1,\, 1)\,, (-1,\, -1)\,, \left(\sqrt{\frac{5}{3}},\, -\sqrt{\frac{5}{3}}\right), \left(-\sqrt{\frac{5}{3}},\, \sqrt{\frac{5}{3}}\right)\right\}$$

Etape 5 - Valeurs et conclusion

$f(1,1) = 4$

$f(-1,-1) = 4$

$f\left(\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}\right) = \frac{40}{3}$

$f\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{40}{3}$

$$\boxed{\text{Finalement : } f_{\min} = 4,\ f_{\max} = \frac{40}{3}.}$$

Etape 6 - Methode alternative

Autre approche : diagonaliser simultanement la contrainte et l'objectif par un changement de base orthogonal.