Corrigé de l'exercice 1

Extrema sous contrainte en dimension 2 - corrige detaille

▶1. A - Facile (contrainte lineaire)

Etape 0 - Rappel

$$f(x,y) = x^2 + y^2 + 4x - 4y - 1$$

$$g(x,y) = -x + y - 6 = 0 \quad \Longleftrightarrow \quad -x + y = 6$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$L(x,y,\lambda) = x^2 + y^2 + 4x - 4y - 1 - \lambda\left(-x + y - 6\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x + 4, \qquad \frac{\partial f}{\partial y} = 2y - 4$$

$$\frac{\partial g}{\partial x} = -1, \qquad \frac{\partial g}{\partial y} = 1$$

$$\nabla f(x,y) = (2x + 4,\, 2y - 4)\,, \qquad \nabla g(x,y) = (-1,\, 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2x + 4 = \lambda \cdot -1 \\ 2y - 4 = \lambda \\ -x + y = 6 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-1\lambda - 4}{2}, \qquad y = \frac{\lambda + 4}{2}$$

$$-1 \cdot \frac{-1\lambda - 4}{2} + \frac{\lambda + 4}{2} = 6$$

$$\frac{(2)\lambda - (-8)}{2} = 6 \Longrightarrow \lambda = 2$$

$$\Longrightarrow (x,y) = (-3,\, 3)$$

Etape 5 - Valeurs et conclusion

$f(-3,3) = -7$

Sur la droite de contrainte, cette valeur est le minimum global ; il n'y a pas de maximum global.

$$\boxed{\text{Finalement : } (x,y) = (-3,3)\,,\ f_{\min} = -7,\ \text{pas de maximum global.}}$$

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = 6 + x$$
$$\phi(x) = f(x, 6+x) = 2x^2 + 12x + 11$$
$$\phi'(x) = 4x + 12 = 0 \Longrightarrow x_* = -3,\ y_* = 3$$
$$\phi(x_*) = f_{\min} = -7$$

▶2. B - Facile (variante lineaire)

Etape 0 - Rappel

$$f(x,y) = x^2 + y^2 - 2x - 4y - 2$$
$$g(x,y) = 2x + y - 9 = 0 \quad \Longleftrightarrow \quad 2x + y = 9$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$
$$L(x,y,\lambda) = x^2 + y^2 - 2x - 4y - 2 - \lambda(2x + y - 9)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x - 2, \qquad \frac{\partial f}{\partial y} = 2y - 4$$
$$\frac{\partial g}{\partial x} = 2, \qquad \frac{\partial g}{\partial y} = 1$$
$$\nabla f(x,y) = (2x-2,\ 2y-4)\,, \qquad \nabla g(x,y) = (2,\ 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} 2x - 2 = \lambda \cdot 2 \\ 2y - 4 = \lambda \\ 2x + y = 9 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{2\lambda + 2}{2}, \qquad y = \frac{\lambda + 4}{2}$$
$$2 \cdot \frac{2\lambda + 2}{2} + \frac{\lambda + 4}{2} = 9$$
$$\frac{(5)\lambda - (-8)}{2} = 9 \Longrightarrow \lambda = 2$$
$$\Longrightarrow (x, y) = (3,\ 3)$$

Etape 5 - Valeurs et conclusion

$$f\left(3, 3\right) = -2$$

Sur la droite de contrainte, cette valeur est le minimum global ; il n'y a pas de maximum global.

$$\boxed{\text{Finalement : } (x, y) = (3, 3)\ ,\ f_{\min} = -2,\ \text{pas de maximum global.}}$$

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = 9 - 2x$$
$$\phi(x) = f\left(x, 9 - 2x\right) = 5x^2 - 30x + 43$$
$$\phi'(x) = 10x - 30 = 0 \Longrightarrow x_* = 3,\ y_* = 3$$
$$\phi(x_*) = f_{\min} = -2$$

▶3. C - Moyen (cercle)

Etape 0 - Rappel

$$f(x, y) = -2x - 2y - 1$$
$$g(x, y) = x^2 + y^2 - 8 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$
$$L(x, y, \lambda) = -2x - 2y - 1 - \lambda\left(x^2 + y^2 - 8\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = -2, \qquad \frac{\partial f}{\partial y} = -2$$
$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = 2y$$
$$\nabla f(x, y) = (-2,\ -2)\,, \qquad \nabla g(x, y) = (2x,\ 2y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} -2 = 2\lambda x \\ -2 = 2\lambda y \\ x^2 + y^2 = 8 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-2}{2\lambda}, \qquad y = \frac{-2}{2\lambda}$$

$$\left(\frac{-2}{2\lambda}\right)^2 + \left(\frac{-2}{2\lambda}\right)^2 = 8$$

$$\frac{8}{4\lambda^2} = 8 \Longrightarrow \lambda^2 = \frac{1}{4} \Longrightarrow \lambda = \pm\frac{1}{2}$$

$$\lambda = \frac{1}{2} \Rightarrow (x,y) = (-2,\ -2), \qquad \lambda = -\frac{1}{2} \Rightarrow (x,y) = (2,\ 2)$$

Etape 5 - Valeurs et conclusion

$$f(-2,-2) = 7$$

$$f(2,2) = -9$$

$$\boxed{\text{Finalement : } f_{\min} = -9 \text{ et } f_{\max} = 7.}$$

Etape 6 - Methode alternative

Parametrisation du cercle.

$$x = \sqrt{8}\cos t, \qquad y = \sqrt{8}\sin t$$

$$f(t) = \sqrt{8}\,(-2\cos t + -2\sin t) - 1$$

$$|-2\cos t + -2\sin t| \leq \sqrt{8} = \sqrt{8}$$

$$\Longrightarrow -9 \leq f(t) \leq 7$$

▶**4. D - Difficile (ellipse generale + quadratique)**

Etape 0 - Rappel

$$f(x,y) = x^2 + y^2$$

$$g(x,y) = 2x^2 + 1xy + 2y^2 - 5 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = x^2 + y^2 - \lambda \left(2x^2 + 1xy + 2y^2 - 5\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = 4x + 1y, \qquad \frac{\partial g}{\partial y} = 1x + 4y$$

$$\nabla f = (2x,\ 2y), \qquad \nabla g = (4x + 1y,\ 1x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 2x = \lambda(4x + 1y) \\ 2y = \lambda(1x + 4y) \\ 2x^2 + 1xy + 2y^2 = 5 \end{cases}$$

Etape 4 - Resolution pas a pas

Somme des deux equations : $2(x + y) = \lambda(4 + 1)(x + y)$

Difference : $2(x - y) = \lambda(4 - 1)(x - y)$

$\Longrightarrow (2 - (4 + 1)\lambda)(x + y) = 0, \quad (2 - (4 - 1)\lambda)(x - y) = 0$

(i) $x = y = t \Rightarrow (4 + 1)t^2 = 5 \Rightarrow t = \pm 1$

(ii) $x = -y = t \Rightarrow (4 - 1)t^2 = 5 \Rightarrow t = \pm\sqrt{\frac{5}{3}}$

$$\mathcal{C} = \left\{ (1,\ 1), (-1,\ -1), \left(\sqrt{\frac{5}{3}},\ -\sqrt{\frac{5}{3}}\right), \left(-\sqrt{\frac{5}{3}},\ \sqrt{\frac{5}{3}}\right) \right\}$$

Etape 5 - Valeurs et conclusion

$$f(1, 1) = 2$$

$$f(-1, -1) = 2$$

$$f\left(\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}\right) = \frac{10}{3}$$

$$f\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{10}{3}$$

$$\boxed{\text{Finalement : } f_{\min} = 2,\ f_{\max} = \frac{10}{3}.}$$

Etape 6 - Methode alternative

Autre approche : diagonaliser la contrainte via u=x+y et v=x-y.

$$2x^2 + 1xy + 2y^2 = \frac{4+1}{4}u^2 + \frac{4-1}{4}v^2, \qquad f = \frac{u^2+v^2}{2}$$

▶5. E - Plus difficile (ellipse generale + quadratique riche)

Etape 0 - Rappel

$$f(x,y) = 3x^2 - 2xy + 3y^2$$

$$g(x,y) = 2x^2 + 1xy + 2y^2 - 5 = 0$$

Etape 1 - Lagrangien

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$L(x,y,\lambda) = 3x^2 - 2xy + 3y^2 - \lambda\left(2x^2 + 1xy + 2y^2 - 5\right)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 6x - 2y, \qquad \frac{\partial f}{\partial y} = -2x + 6y$$

$$\frac{\partial g}{\partial x} = 4x + 1y, \qquad \frac{\partial g}{\partial y} = 1x + 4y$$

$$\nabla f = (6x - 2y,\ -2x + 6y), \qquad \nabla g = (4x + 1y,\ 1x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 6x - 2y = \lambda(4x + 1y) \\ -2x + 6y = \lambda(1x + 4y) \\ 2x^2 + 1xy + 2y^2 = 5 \end{cases}$$

Etape 4 - Resolution pas a pas

Somme des deux equations : $4(x+y) = \lambda(4+1)(x+y)$

Difference : $8(x-y) = \lambda(4-1)(x-y)$

$\Longrightarrow (4 - (4+1)\lambda)(x+y) = 0, \quad (8 - (4-1)\lambda)(x-y) = 0$

(i) $x = y = t \Rightarrow (4+1)t^2 = 5 \Rightarrow t = \pm 1$

(ii) $x = -y = t \Rightarrow (4-1)t^2 = 5 \Rightarrow t = \pm\sqrt{\frac{5}{3}}$

$$\mathcal{C} = \left\{(1,\ 1), (-1,\ -1), \left(\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}\right), \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)\right\}$$

Etape 5 - Valeurs et conclusion

$f(1,1) = 4$

$f(-1,-1) = 4$

$f\left(\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}\right) = \frac{40}{3}$

$f\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{40}{3}$

$\boxed{\text{Finalement : } f_{\min} = 4,\ f_{\max} = \frac{40}{3}.}$

Etape 6 - Methode alternative

Autre approche : diagonaliser simultanement la contrainte et l'objectif par un changement de base orthogonal.