

Corrigé de l'exercice 1**Extrema sous contrainte en dimension 2 - corrige detaille****►1. A - Facile (contrainte lineaire)****Etape 0 - Rappel**

$$f(x, y) = x^2 + y^2 - 2x - 2y$$

$$g(x, y) = -3x + y - 8 = 0 \iff -3x + y = 8$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = x^2 + y^2 - 2x - 2y - \lambda(-3x + y - 8)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x - 2, \quad \frac{\partial f}{\partial y} = 2y - 2$$

$$\frac{\partial g}{\partial x} = -3, \quad \frac{\partial g}{\partial y} = 1$$

$$\nabla f(x, y) = (2x - 2, 2y - 2), \quad \nabla g(x, y) = (-3, 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x, y) = 0 \end{cases}$$

$$\begin{cases} 2x - 2 = \lambda \cdot -3 \\ 2y - 2 = \lambda \\ -3x + y = 8 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-3\lambda + 2}{2}, \quad y = \frac{\lambda + 2}{2}$$

$$-3 \cdot \frac{-3\lambda + 2}{2} + \frac{\lambda + 2}{2} = 8$$

$$\frac{(10)\lambda - (4)}{2} = 8 \implies \lambda = 2$$

$$\implies (x, y) = (-2, 2)$$

Etape 5 - Valeurs et conclusion

$$f(-2, 2) = 8$$

Sur la droite de contrainte, cette valeur est le minimum global; il n'y a pas de maximum global.

Enfinement : $(x, y) = (-2, 2)$, $f_{\min} = 8$, pas de maximum global.

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = 8 + 3x$$

$$\phi(x) = f(x, 8 + 3x) = 10x^2 + 40x + 48$$

$$\phi'(x) = 20x + 40 = 0 \implies x_* = -2, y_* = 2$$

$$\phi(x_*) = f_{\min} = 8$$

►2. B - Facile (variante lineaire)

Etape 0 - Rappel

$$f(x, y) = x^2 + y^2 - 2x + 4y + 2$$

$$g(x, y) = -x + y + 1 = 0 \iff -x + y = -1$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = x^2 + y^2 - 2x + 4y + 2 - \lambda(-x + y + 1)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 2x - 2, \quad \frac{\partial f}{\partial y} = 2y + 4$$

$$\frac{\partial g}{\partial x} = -1, \quad \frac{\partial g}{\partial y} = 1$$

$$\nabla f(x, y) = (2x - 2, 2y + 4), \quad \nabla g(x, y) = (-1, 1)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x, y) = 0 \end{cases}$$

$$\begin{cases} 2x - 2 = \lambda \cdot -1 \\ 2y + 4 = \lambda \\ -x + y = -1 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{-1\lambda + 2}{2}, \quad y = \frac{\lambda - 4}{2}$$

$$-1 \cdot \frac{-1\lambda + 2}{2} + \frac{\lambda - 4}{2} = -1$$

$$\frac{(2)\lambda - (6)}{2} = -1 \implies \lambda = 2$$

$$\implies (x, y) = (0, -1)$$

Etape 5 - Valeurs et conclusion

$$f(0, -1) = -1$$

Sur la droite de contrainte, cette valeur est le minimum global; il n'y a pas de maximum global.

Finalemment : $(x, y) = (0, -1)$, $f_{\min} = -1$, pas de maximum global.

Etape 6 - Methode alternative

Substitution (toujours possible pour une contrainte lineaire).

$$y = -1 + x$$

$$\phi(x) = f(x, -1 + x) = 2x^2 - 1$$

$$\phi'(x) = 4x = 0 \implies x_* = 0, \quad y_* = -1$$

$$\phi(x_*) = f_{\min} = -1$$

►3. C - Moyen (cercle)**Etape 0 - Rappel**

$$f(x, y) = x + 3y$$

$$g(x, y) = x^2 + y^2 - 10 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = x + 3y - \lambda (x^2 + y^2 - 10)$$

Etape 2 - Gradients detailles

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 3$$

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y$$

$$\nabla f(x, y) = (1, 3), \quad \nabla g(x, y) = (2x, 2y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x, y) = 0 \end{cases}$$

$$\begin{cases} 1 = 2\lambda x \\ 3 = 2\lambda y \\ x^2 + y^2 = 10 \end{cases}$$

Etape 4 - Resolution pas a pas

$$x = \frac{1}{2\lambda}, \quad y = \frac{3}{2\lambda}$$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$$

$$\frac{10}{4\lambda^2} = 10 \implies \lambda^2 = \frac{1}{4} \implies \lambda = \pm \frac{1}{2}$$

$$\lambda = \frac{1}{2} \implies (x, y) = (1, 3), \quad \lambda = -\frac{1}{2} \implies (x, y) = (-1, -3)$$

Etape 5 - Valeurs et conclusion

$$f(1, 3) = 10$$

$$f(-1, -3) = -10$$

Finalemment : $f_{\min} = -10$ et $f_{\max} = 10$.

Etape 6 - Methode alternative

Parametrisation du cercle.

$$x = \sqrt{10} \cos t, \quad y = \sqrt{10} \sin t$$

$$f(t) = \sqrt{10} (1 \cos t + 3 \sin t)$$

$$|1 \cos t + 3 \sin t| \leq \sqrt{10} = \sqrt{10}$$

$$\implies -10 \leq f(t) \leq 10$$

►4. D - Difficile (ellipse generale + quadratique)**Etape 0 - Rappel**

$$f(x, y) = x^2 + y^2$$

$$g(x, y) = 2x^2 + 2xy + 2y^2 - 6 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = x^2 + y^2 - \lambda (2x^2 + 2xy + 2y^2 - 6)$$

Etape 2 - Gradients detaillés

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = 4x + 2y, \quad \frac{\partial g}{\partial y} = 2x + 4y$$

$$\nabla f = (2x, 2y), \quad \nabla g = (4x + 2y, 2x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 2x = \lambda(4x + 2y) \\ 2y = \lambda(2x + 4y) \\ 2x^2 + 2xy + 2y^2 = 6 \end{cases}$$

Etape 4 - Resolution pas a pas

Somme des deux equations : $2(x + y) = \lambda(4 + 2)(x + y)$

Difference : $2(x - y) = \lambda(4 - 2)(x - y)$

$$\implies (2 - (4 + 2)\lambda)(x + y) = 0, \quad (2 - (4 - 2)\lambda)(x - y) = 0$$

(i) $x = y = t \implies (4 + 2)t^2 = 6 \implies t = \pm 1$

(ii) $x = -y = t \implies (4 - 2)t^2 = 6 \implies t = \pm \sqrt{\frac{6}{2}}$

$$C = \left\{ (1, 1), (-1, -1), \left(\sqrt{\frac{6}{2}}, -\sqrt{\frac{6}{2}} \right), \left(-\sqrt{\frac{6}{2}}, \sqrt{\frac{6}{2}} \right) \right\}$$

Etape 5 - Valeurs et conclusion

$$f(1, 1) = 2$$

$$f(-1, -1) = 2$$

$$f\left(\sqrt{\frac{6}{2}}, -\sqrt{\frac{6}{2}}\right) = 6$$

$$f\left(-\sqrt{\frac{6}{2}}, \sqrt{\frac{6}{2}}\right) = 6$$

Finalement : $f_{\min} = 2, f_{\max} = 6.$
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Etape 6 - Methode alternative

Autre approche : diagonaliser la contrainte via $u=x+y$ et $v=x-y$.

$$2x^2 + 2xy + 2y^2 = \frac{4+2}{4}u^2 + \frac{4-2}{4}v^2, \quad f = \frac{u^2 + v^2}{2}$$

►5. E - Plus difficile (ellipse generale + quadratique riche)

Etape 0 - Rappel

$$f(x, y) = 3x^2 - 2xy + 3y^2$$

$$g(x, y) = 2x^2 + 2xy + 2y^2 - 6 = 0$$

Etape 1 - Lagrangien

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$L(x, y, \lambda) = 3x^2 - 2xy + 3y^2 - \lambda(2x^2 + 2xy + 2y^2 - 6)$$

Etape 2 - Gradients detaillés

$$\frac{\partial f}{\partial x} = 6x - 2y, \quad \frac{\partial f}{\partial y} = -2x + 6y$$

$$\frac{\partial g}{\partial x} = 4x + 2y, \quad \frac{\partial g}{\partial y} = 2x + 4y$$

$$\nabla f = (6x - 2y, -2x + 6y), \quad \nabla g = (4x + 2y, 2x + 4y)$$

Etape 3 - Systeme de Lagrange

$$\begin{cases} 6x - 2y = \lambda(4x + 2y) \\ -2x + 6y = \lambda(2x + 4y) \\ 2x^2 + 2xy + 2y^2 = 6 \end{cases}$$

Etape 4 - Resolution pas a pas

$$\text{Somme des deux equations : } 4(x + y) = \lambda(4 + 2)(x + y)$$

$$\text{Difference : } 8(x - y) = \lambda(4 - 2)(x - y)$$

$$\implies (4 - (4 + 2)\lambda)(x + y) = 0, \quad (8 - (4 - 2)\lambda)(x - y) = 0$$

$$(i) \quad x = y = t \implies (4 + 2)t^2 = 6 \implies t = \pm 1$$

$$(ii) \quad x = -y = t \implies (4 - 2)t^2 = 6 \implies t = \pm \sqrt{\frac{6}{2}}$$

$$\mathcal{C} = \left\{ (1, 1), (-1, -1), \left(\sqrt{\frac{6}{2}}, -\sqrt{\frac{6}{2}} \right), \left(-\sqrt{\frac{6}{2}}, \sqrt{\frac{6}{2}} \right) \right\}$$

Etape 5 - Valeurs et conclusion

$$f(1, 1) = 4$$

$$f(-1, -1) = 4$$

$$f\left(\sqrt{\frac{6}{2}}, -\sqrt{\frac{6}{2}}\right) = 24$$

$$f\left(-\sqrt{\frac{6}{2}}, \sqrt{\frac{6}{2}}\right) = 24$$

Finalemment : $f_{\min} = 4$, $f_{\max} = 24$.
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Etape 6 - Methode alternative

Autre approche : diagonaliser simultanément la contrainte et l'objectif par un changement de base orthogonal.